Reoptimizations in linear programming

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REOPTIMIZATIONS IN LINEAR PROGRAMMING

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Abstract: *Replacing a real process which we are concerned in with other more convenient for the study is called modeling. After the replacement, the model is analyzed and the results we get are expanded on that process. Mathematical models being more abstract, they are also more general and so, more important. Mathematical programming is known as analysis of various concepts of economic activities with the help of mathematical models.*

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To solve management problems requires simplified representation of reality through models. A model is a simplified selective rendering of a real-life system [1].

The real system that a model describes can be already existing or may be an idea conceived in the pending execution. In the first case, the design objective is to provide means to analyze the system behavior to improve its performance. In the second case, the design objective is to define the ideal structure of the future system.

The real system can be extremely complicated and the model may not faithfully represent all information in reality. This is practically impossible.

In developing a quantitative model, represented by mathematical relationships and symbols, there must be taken into consideration two conditions:

- the model must achieve its purpose;
- the model can be solved in the process affected time.

Among the most common models in the economic practice are optimization models, those which are concerned with maximizing (minimizing) of an objective function. A linear programming problem is a special case of the general problem of optimization, where objective function and the constraints are linear[1]. Inequalities and the non-negativity restrictions placed on variables are not only permitted, but are typical. The importance of linear programming consists precisely in the ability to solve such problems [5].

1. Mathematical model of a linear programming problem

The ***mathematical programming problem*** usually means determining the ***optimal value*** (***minimum*** or ***maximum***) of a function of several variables (also called ***objective function***, ***purpose function*** or ***efficiency function***), if variables are subject to some ties expressed by equations and inequations (also called ***restrictions***) and to conditions on the sign. If the objective function and constraints are linear functions, the problem is called ***linear programming problem***.

[1] A similar presentation of linear programming can be found in the work of D. Gale *The Theory of Linear Economic Models,* McGraw-Hill, 1960.

The general form of a linear programming problem is:

$$\min(\max) f = \sum_{j=1}^{n} c_j x_j \tag{1}$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i,\ i = \overline{1,k} \tag{2}$$

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i,\ i = \overline{k+1,p} \tag{3}$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i,\ i = \overline{p+1,m} \tag{4}$$

$$x_j \geq 0,\ j = \overline{1,n} \tag{5}$$

Definition 1. Function (1) is called the ***objective function*** (***purpose function*** or ***function of efficiency***) of the linear programming problem.

Definition 2. Conditions (2), (3) and (4) are called ***restrictions*** of linear programming problem.

Definition 3. The vector $x = (x_1, x_2, \ldots, x_n)^t$ which checks the restrictions (2), (3) and (4) is called ***possible solution*** of the linear programming problem.

Definition 4. A possible solution of the linear programming problem which also checks the conditions (5) is called admissible.

Definition 5. The admissible solution for which it is obtained the minimum or maximum value of the function (1) is called the ***optimal solution*** of the problem.

Observation 6. Conditions (5) are also called non-negativity conditions.

If we note the vector of the unknowns by $x = (x_1, x_2, \ldots, x_n)^t$, that of the coefficients by $c = (c_1, c_2, \ldots, c_m)$, the matrix of the coefficients by $A = \left(a_{ij}\right)_{\substack{i=\overline{1,m} \\ j=\overline{1,n}}}$, and the terms vector from the right member of the restrictions by $b = (b_1, b_2, \ldots, b_m)^t$, we get:

1. **The matrix form of linear programming problem:**

$$\min(\max) f(x) = cx$$
$$Ax \leq b\ (Ax \geq b)$$
$$x \geq 0$$

2. **The canonical form of linear programming problem:**

(i) for the problem of minimum

$$\min\ f(x) = cx$$
$$Ax \geq b$$
$$x \geq 0$$

(ii) for the problem of maximum

$$\max\ f(x) = cx$$
$$Ax \leq b$$
$$x \geq 0$$

3. **The standard form of linear programming problem:**

$$\min(\max) f(x) = cx$$
$$Ax = b$$
$$x \geq 0$$

2. Classification of solutions of a linear programming problem

Let be the linear programming problem as standard:

$$\begin{aligned} &\min(\max) f(x) = cx \\ &Ax = b \qquad\qquad (6) \\ &x \geq 0 \end{aligned}$$

Let us note by A the set of admissible solutions (see Definition 4). If A is different from the empty set, then the linear programming problem has finite optimum, if $A \neq \Phi$ and f infinite, the linear programming problem has infinite optimum, and if $A = \Phi$, then the linear programming problem is incompatible.

Definition 7. A possible solution for which the number of nonzero components is less than or equal to the number of restrictions, and the column vectors of matrix A which correspond to these components are linearly independent, is called the ***basic solution*** of linear programming problems.

Definition 8. A possible solution is called ***nondegenerated*** if all its components are nonzero and degenerated otherwise.

Definition 9. A possible solution that achieves the optimum of the objective function is called an ***optimal solution*** of the linear programming problem.

Theorem 10. The set of solutions of linear programming problem is linearly convex.

Proof:

Let be x_1 and x_2 solutions of problem (6), which means $x_1 \geq 0, x_2 \geq 0$, $Ax_1 = b$, $Ax_2 = b$ and let be $x = (1-\lambda)x_1 + \lambda x_2, \lambda \in [0,1]$ their convex combination. Because $1-\lambda \geq 0$ and $\lambda \geq 0$ it results $x \geq 0$.

$$Ax = A[(1-\lambda)x_1 + \lambda x_2] = Ax_1 - \lambda(Ax_1 - Ax_2) = Ax_1 = b$$

which means that the x vector is still a solution, and so the set of solutions is convex.

Theorem 11 [5]. The set of optimal solutions of a linear programming problem is convex.

Proof:

We take for example the case of the linear programming problem:

$$\begin{aligned} &\min f(x) = cx \\ &Ax = b \\ &x \geq 0 \end{aligned}$$

Let be x_1 and x_2 optimal solutions of the problem, which means $\min f(x_1) = \min f(x_2)$.

It results $\min f(x_1) = \min f(x)$, where $x = (1-\lambda)x_1 + \lambda x_2, \lambda \in [0,1]$, $x \geq 0$.

Analog is the case of the linear programming problem of maximum.

Observation 12. An admissible solution which is not a linear convex combination of other admissible solutions is a "peak" of the set of admissible solutions (the convex polyhedron).

Theorem 13. If the linear programming problem (6) allows optimal solutions, then there is at least one peak of the admissible solutions set that is optimal solution.

Proof:

Let be x_1 and x_2 the peaks of the admissible solutions set and let be $\tilde{x}$ a vector of this set, representing an optimal solution. Supposing $\tilde{x}$ is not a peak, it can be written as a convex linear combination of the peaks:

$$\tilde{x} = \sum_{i=1}^{k} \lambda_i x_i, \lambda_i \geq 0, i = \overline{1,k}\ ;\ \sum_{i=1}^{k} \lambda_i = 1.$$

So $f(\tilde{x}) = c\tilde{x} = c\sum_{i=1}^{k} \lambda_i f(x_i)$.

Assuming that ascending ordering numbers $f(x_i), i=\overline{1,k}$ there have been obtained the inequalities:

$$f(x_1) \le f(x_2) \le \ldots \le f(x_k)$$

Then

$$f(x_1)\sum_{i=1}^{k}\lambda_i \le \sum_{i=1}^{k}\lambda_i f(x_i) \le f(x_k)\sum_{i=1}^{k}\lambda_i$$

that is

$$f(x_1) \le f(\tilde{x}) \le f(x_k).$$

which means that also x_1 (which is a peak of admissible solution set) is an optimal solution of the problem (6).

Theorem 14. If $\tilde{x}$ is a basic optimal solution of linear programming problem (6), then the vectors which correspond to nonzero components of *x* are linearly independent.

<u>*Proof:*</u>

Supposing that $rangA = m$ and $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_m, 0, \ldots, 0)^t$, $\tilde{x}_1 > 0, \tilde{x}_2 > 0, \ldots, \tilde{x}_m > 0$. If $v_j = (a_{1j}, a_{2j}, \ldots, a_{mj})^t$, $j = \overline{1,n}$ are column vectors of the *A* matrix, then $v_1, v_2, \ldots, v_m$ correspond to *m* nonzero components of $\tilde{x}$ and out of the condition $A\tilde{x} = b$ results:

$$\tilde{x}_1 v_1 + \tilde{x}_2 v_2 + \ldots + \tilde{x}_m v_m = b \qquad (7)$$

Let's assume against all reason that the vectors $v_1, v_2, \ldots, v_m$ are linearly dependent, namely there are constants $\alpha_1, \alpha_2, \ldots, \alpha_m$ not all zero, so that:

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m = 0 \qquad (8)$$

If we multiply by β relation (8) and add at (7), we obtained:

$$(\tilde{x}_1 + \alpha_1 \beta) v_1 + (\tilde{x}_2 + \alpha_2 \beta) v_2 + \ldots + (\tilde{x}_m + \alpha_m \beta) v_m = b \qquad (9)$$

If we multiply then by β relation (8) and decrease of (7), we obtained:

$$(\tilde{x}_1 - \alpha_1 \beta) v_1 + (\tilde{x}_2 - \alpha_2 \beta) v_2 + \ldots + (\tilde{x}_m - \alpha_m \beta) v_m = b \qquad (10)$$

Chossing for β the value β^* for which $\tilde{x}_j + \beta^* \alpha_j > 0$, $\tilde{x}_j - \beta^* \alpha_j > 0$, $j = \overline{1,m}$, the equalities (9) and (10) define two admissible solutions:

$$x_1 = (\tilde{x}_1 + \beta^* \alpha_1, \tilde{x}_2 + \beta^* \alpha_2, \ldots, \tilde{x}_m + \beta^* \alpha_m, 0, \ldots, 0)^t$$

$$x_2 = (\tilde{x}_1 - \beta^* \alpha_1, \tilde{x}_2 - \beta^* \alpha_2, \ldots, \tilde{x}_m - \beta^* \alpha_m, 0, \ldots, 0)^t$$

with the property $\tilde{x} = \frac{1}{2}(x_1 + x_2)$. This equality expresses the fact that the optimal solution x is a linear combination of two vectors in the set of admissible solutions, so it is not a peak. Absurd, because according to theorem 13, an optimal solution is a peak of the admissible solutions set.

Therefore, the vectors $v_1, v_2, \ldots, v_m$ are linearly independent, so they form a basis.

3. Primal simplex algorithm

The general method for solving linear programming problem is known as the method (algorithm) simplex. The name of the simplex method, first published in 1947 by the American mathematician G.B.Dantzig, is linked to the fact that a convex polyhedron is also called simplex [1].

The simplex method has the advantage that besides a relatively small calculations volume, can give accurate and conclusive answers to all situations that can be presented by solving a linear programming problem.

It is an effective method for solving linear programming, which allows us to systematically explore the set of basic solutions. The simplex method consists of a sequence of iterations, each iteration determining a basic solution with the feature that the objective function value is continuously improved up to the optimum, if any. The principle of work is changing the base, a method presented in the previous chapter (the pivot method). The question however is which vector enters the base and which one is leaving, to ensure the search on the set of basic admissible solutions.

Given linear programming problem as standard (6):

$$\min(\max) f(x) = cx$$
$$Ax = b$$
$$x \geq 0$$

Assuming that the columns $a_1, a_2, ..., a_n$ of matrix A have indices $j = 1, ..., n$ belonging to the set J_B or J_S of indices, as the corresponding variables are basic variables, or secondary variables. So,

$$J_B \cup J_S = \{1, 2, ..., n\}, J_B \cap J_S = \Phi$$

Let B a base consisting of m columns of A.

The system $Ax = b$ becomes

$Bx^B + Sx^S = b$, whence, multiplying to the left with B^{-1} we get

$x^B = B^{-1}b - B^{-1}Sx^S$.

We note $\overline{x}^B = B^{-1}b; y_j^B = B^{-1}a_j, j \in J_S$.

Then $x^B = \overline{x}^B - \sum_{j \in J_S} y_j^B x_j$,

or on components $x_i = \overline{x}_i^B - \sum_{j \in J_S} y_{ij}^B x_j, i \in J_B$.

A basic solution can be obtained for $x^S = 0$, so $\overline{x}^B = B^{-1}b$.

A basic solution $\overline{x}^B = B^{-1}b$ is admissible if $\overline{x}^B \geq 0$.

A base B which checks such a condition is called primal admissible base.

We express the objective function with the help of secondary variables.

Writing adequately the c vector of the function coefficients, we obtain:

$$f = cx = c^B x^B + c^S x^S = \sum_{i \in J_B} c_i x_i + \sum_{j \in J_S} c_j x_j =$$
$$= \sum_{i \in J_B} c_i \left(\overline{x}_i^B - \sum_{j \in J_S} y_{ij}^B x_j \right) + \sum_{j \in J_S} c_j x_j =$$
$$= \sum_{i \in J_B} c_i \overline{x}_i^B - \sum_{j \in J_S} \left(-c_j + \sum_{i \in J_B} c_i y_{ij}^B \right) x_j .$$

We note $\overline{z}^B = \sum_{i \in J_B} c_i \overline{x}_i^B = c^B \overline{x}^B$ and $z_j^B = \sum_{i \in J_B} c_i y_{ij}^B = c^B y_j^B, j \in J_S$

$$f = \overline{z}^B - \sum_{j \in J_S} (z_j^B - c_j) x_j .$$

Theorem 15. If B is a primal admissible base and for any $j \in J_S$ we have $z_j^B - c_j \leq 0$, then the corresponding basic programme of base B ($\overline{x}^B = B^{-1}b, x^S = 0$) is an optimal programme for problem (6).

<u>*Proof:*</u>

Let be $x \geq 0$ a certain programme. From the theorem's hypothesis we have $\sum_{j \in J_S} (z_j^B - c_j) x_j \leq 0$.

As $f = \overline{z}^B - \sum_{j \in J_S} \left(z_j^B - c_j\right) x_j$ we obtain $f \geq \overline{z}^B$ so x^B and x^S represents the optimal programme for problem (6).

Theorem 16. If for a primal admissible base *B* take place the following conditions:

i) There $k \in J_S$ is so that $z_k^B - c_k > 0$

ii) The basic programme $\overline{x}^B = B^{-1}b, x^S = 0$ is nondegenerated, so the basic programme corresponding to *B* is not optimal.

<u>*Proof:*</u>

Let x' be a programme with the property that the variables'values $x_i, i = \overline{1,n}$ are the same to those corresponding to base *B*, less the variable x_k, which takes a value low enough x_k^0 so that the problem's conditions to be satisfied.

As $x_i^B = \overline{x}_i^B - \sum_{j \in J_S} y_{ij}^B x_j$ it results that is enough that $\overline{x}_i^B - y_{ik}^B x_k^0 \geq 0$, $(\forall) i \in J_B$.

Noting by f' the objective function value corresponding to x', we get from $f = \overline{z}^B - \sum_{j \in J_S} \left(z_j^B - c_j\right) x_j$ the following:

$f' = \overline{z}^B - \left(z_k^B - c_k\right) x_k^0 < \overline{z}^B$, so the basic programme corresponding to *B* is not optimal.

Theorem 17. If for an admissible primal base *B* take place the following conditions:

i) There is $k \in J_S$ so that $z_k^B - c_k > 0$;

ii) The basic programme $\overline{x}^B = B^{-1}b, x^S = 0$ in nondegenerated;

iii) $y_{ik} \leq 0, (\forall) i \in J_B$, so the problem has the infinite optimum

<u>*Proof:*</u>

The hypothesis (i) and (ii) being the same with those of the previous theorem, it is noticed that the vector x' defined above stays programme of the problem for any x_k value, the value of objective function f' given by:

$f' = \overline{z}^B - \left(z_k^B - c_k\right) x_k^0 < \overline{z}^B$

becomes very small for x_k^0 chosen very big and so the limit of f' is $-\infty$.

Theorem 18. If for an admissible primal base *B* take place the following conditions:

(i) There is $k \in J_S$ so that $z_k^B - c_k > 0$;

(ii) The basic programme $\overline{x}^B = B^{-1}b, x^S = 0$ is nondegenerated;

(iii) $(\exists) i \in J_B$ so that $y_{ik} > 0$

then the maximum value that we can assign to x^0 so that x' to stay programme is given by:

$$\theta_{\min} = \min_{\substack{i \in J_B \\ y_{ik} > 0}} \left(\frac{\overline{x}_i^B}{y_{ik}^B} \right) = \frac{\overline{x}_r^B}{y_{rk}^B}.$$

If we assign to x_k^0 this value, then the corresponding programme of x is a basic solution. This corresponds to a *B'* base which it is obtained from *B* by replacing the column a_r by column a_k .

Proof:

Let $J_B^+ = \{ i \in J_B / y_{ik} > 0 \}$.

If $x_k \leq \frac{x_i^B}{y_{ik}^B}, (\forall) i \in J_B^+$ then the condition $\overline{x}_i^B - y_{ik}^B x_k^0 \geq 0$, $(\forall) i \in J_B$ is fulfilled.

Let us assume that we give to x_k the value $\theta_{\min}$. From $x^B = \overline{x}^B - \sum_{j \in J_S} y_j^B x_j$ results that $x_r = 0$.

Therefore, we obtain a new basic solution made of $x_i, i \in I_B - \{r\}$ and x_k .

The *B'* base corresponding to this one is obtained from *B* by replacing column a_r by column a_k . From the fact that $y_{rk} \neq 0$ results that the column vectors of *B'* are linearly independent.

Observation 19. According to formula $f' = \overline{z}^B - (z_k^B - c_k) x_k^0$ the objective function value corresponding to base *B'* is

$$\overline{z}^{B'} = \overline{z}^B - (z_k^B - c_k) \frac{\overline{x}_r}{y_{rk}} \tag{11}$$

If there are multiple indices *k* with the property $z_k - c_k > 0$ then, to get the lowest objective function value, it should be chosen that index for which the quantity to be deducted in relation (11) to be maximum. Since the calculations are quite laborious, in practice is chosen that index which maximizes the expression $z_j^B - c_j$.

Case 1. There is initial basis:

Let linear programming problem be the general matrix form:

$$\min (\max) f(x) = cx$$
$$Ax \leq b$$
$$x \geq 0$$

Phase I:

The linear programming problem is brought in standard form. Auxiliary variables y_i are introduced in the objective function with the cost 0.

The extended matrix of linear programming problem (of the standard form) will be:

$$\overline{A} = (A, I) = \begin{pmatrix} a_{11} & a_{12} & \ldots a_{1n} & 1 & \ldots 0 \\ a_{m1} & a_{m2} & \ldots a_{mn} & 0 & \ldots 1 \end{pmatrix}$$

$\Rightarrow$ There is initial base given by the vectors y_i.

Phase II:

The simplex table it is built as follows:

B	C_B	b	c_1 c_2 ... c_n x_1 x_2 ... x_n	0 ... 0 y_1 ... y_m	$\theta_{\min}$
y_1 y_2 ... y_m	0 0 ... 0	b_1 b_2 ... b_m	A	I	
	Δ_j		$-c_1$ $-c_2$ ... $-c_n$	0 ...0	

Values $\Delta_j = z_j^B - c_j$ are calculated using the formula:

$\Delta_j = z_j^B - c_j = C_B \cdot a_j - c_j$, where a_j is the corresponding column of the vector x_j.

The optimum criteria:

If the problem is of maximum and all $\Delta_j \geq 0 \Rightarrow$ the solution is optimal.

If the problem is of minimum and all $\Delta_j \leq 0 \Rightarrow$ the solution is optimal.

The solution is read on the column of the vector *b*. In case the requirements for optimal criteria is not met, then it is applied the entry criteria.

The entry criteria:

For linear programming problem of maximum enters in base the vector a_k, corresponding to the difference Δ_j <0, the largest in absolute value (or the smallest negative one).

For linear programming problem of minimum enters in base the vector a_k, corresponding to the difference Δ_j >0, the largest one.

The exit criteria:

Vector a_k corresponding to the report

$$\theta_{\min} = \min_i \{\frac{b_i}{a_{ik}}, a_{ik} > 0\}$$

is leaving the base.

At the intersection of the output line with the input column there is the pivot. The pivot method applies and then it is continued the process until the optimal solution it is obtained.

Observation 20. For the primal simplex algorithm, the pivot is always strictly larger than zero.

Case 2. There is no initial base

It is built an artificial base as follows:

It is written the matrix *A* of the linear programming problem in standard form and where there is no unit vector, there are added to the restrictions the artificial variables u_j. These variables are inserted in the objective function with a very high *M* penalty (positive). In problems of maximum *M* is inserted with the "-" sign, and in problems of minimum *M* is inserted with "+" sign.

Observation 21.

(i) The problem is well formulated if in the optimal solution the artificial variables disappear from the base (are out of the base).

(ii) If an artificial variable is out of the base, it cannot be reintroduced. Its corresponding column in the table is shading.

(iii) This method is also called "the penalty method".

4. Dual simplex algorithm

Dual simplex algorithm is an algorithm for solving primal linear programming problem, which reflects the application of simplex algorithm to the dual problem, sometimes simpler than the considered primal problem. Dual simplex algorithm is in a certain way, dual to the common simplex algorithm.

Definition 22. A solution $\widetilde{x}$ is dual realizable to the primal linear programming problem if:

(i) is optimal - if $\Delta_j \geq 0$ for the maximum linear programming problem

- if $\Delta_j \leq 0$ for the minimum linear programming problem

(ii) is not admissible (on the *b* vector column there are negative elements)

The stages of dual simplex algorithm

The problem is prepared to be solved with dual simplex algorithm by multiplying by (-1) the restrictions of type "≥".

When all the restrictions are of type "≤" we switch to i).

If a restriction is of type "=", then it becomes double inequality.

i) The linear programming problem it is brought to a standard form, and it is built an initial basis, either:
   - only with auxiliary variable y
   - and by adding artificial variables u

ii) We make up the initial simplex table as the primal simplex algorithm. We calculate the Δ_j value. We check if the solution found is dual realizable (it is optimal, but it is not admissible).

iii) If the solution in the initial table is dual realizable, it is applied the dual simplex algorithm.

   a) The exit criteria:
   Within dual simplex algorithm leaves the basis the vector a_r corresponding to the larger negative component in absolute value from the solution found in the initial table corresponding to column *b*.

   b) We calculate the indicators $\gamma = \min(z_j - c_j) \cdot a_{rj},\ a_{rj} < 0$
   We divide the elements from line Δ_j to the strictly negative components of the vector line which is leaving and we choose the minimum.

iv) We continue with the pivot method, and the algorithm is developed until the solution is optimal and admissible.

Observation 23. The pivot in the case of dual simplex algorithm is always negative.

5. Reoptimization in linear programming

In linear programming problems occur sizes that we have considered constant. These are the elements of vectors *b*, *c* and of matrix *A*. However, in the economic activity are frequent changes of the conditions underlying the linear programming models, where it appears the need to develop mathematical methods to allow a dynamic approach of economic phenomena.

Reoptimization consists in recalculating the optimal solution in the situation in which are modified some of the elements of the vectors *b*, *c* and matrix *A*, which shape the activity of a trader.

Let be the linear programming problem given under a standard form (6)

$$\min(\max) f(x) = cx$$
$$Ax = b$$
$$x \geq 0$$

We consider that the initial problem has been solved, so it was obtained the last simplex table with optimal solution $\widetilde{x}^B$.

5.1 Modification of free terms vector of restrictions

With the modification of b vector, it modifies also the optimal solution becoming $\tilde{x}'^B = \tilde{B}^{-1} b'$. But the solution remains optimal, as $z_j^B - c_j = c_B \cdot \tilde{B}^{-1} \cdot a_j - c_j$.

So, from the last simple table of the initial problem is taken the matrix $\tilde{B}^{-1}$ and it is calculated the new solution $\tilde{x}'^B = \tilde{B}^{-1} b'$, existing two possibilities:

a) If all components of the solution $\tilde{x}'^B$ are nonnegative ($\tilde{x}'^B \geq 0$), then this is admissible and as it verifies also the condition of optimum, it results that it will be the new optimal solution.

b) If $\tilde{x}'^B$ are negative components, then the solution is not admissible and it will be a basic solution dual feasible. In this case, it is taken the last simplex table of the initial linear programming problem, replacing $\tilde{x}^B$ with $\tilde{x}'^B$ in column b, and it is continued with dual simplex algorithm.

5.2 Modification of objective function coefficients

It is noted that the basic solution of the initial problem $\tilde{x}^B = \tilde{B}^{-1} b$ does not changes, but the optim indicators change, so that instead of

$$z_j^B - c_j = c_B \cdot \tilde{B}^{-1} \cdot a_j - c_j$$

we have

$$z_j'^B - c_j' = c_B' \cdot \tilde{B}^{-1} \cdot a_j - c_j'$$

Therefore, we resume the last simplex table of the initial problem where there are made the following changes:

i) c_j with c_j' (the coefficients of the above variables x_j from the first line of the table),

ii) column c_B with column c_B'

There are calculated the new differences $z_j'^B - c_j'$. There are the cases:

i) If all differences $z_j'^B - c_j'$ correspond to the optim condition, then the optimal programme remains the same..

ii) If the differences $z_j'^B - c_j'$ does not verify the optim condition then it is applied the primal simplex algorithm until the optimal solution it is obtained.

5.3 Adding column vectors to the matrix of restrictions

Suppose that a number of p variables $x_{n+1}, \ldots, x_{n+p}$ is added with the corresponding vectors a $a_{n+1}, \ldots, a_{n+p}$.

The vectors $a_{n+1}, \ldots, a_{n+p}$ are written in the optimal basis from the last simplex table of the initial linear programming problem, using the formula

$y_j^B = \widetilde{B}^{-1} \cdot a_j$, $j = \overline{n+1, n+p}$

It starts from the last simplex table of the initial linear programming problem and extends to the right with columns of new variables $y_{n+1}^B, \dots, y_{n+p}^B$. Next, it is calculated $z_j^B - c_j$, $j = \overline{n+1, n+p}$, corresponding to the inserted columns, being possible the following situations:

a) If all values $z_j^B - c_j$ for the new added vectors correspond to the optim criteria, then the optimal programme is the same

b) If differences $z_j^B - c_j$ do not verify the optim condition, then the primal simplex algorithm it is applied until it is reached the optimal solution.

Application. A company has at the 4 departments S_1, S_2, S_3, S_4 a disposable of mechanical type of 1200, 1400, 2000, and respectively 800 u.m. Three products P_1, P_2, P_3 are manufactured with the unitar consumption of hours/ machine given in the table.

a) Determine the levels to be produced P_1, P_2, P_3 so that the total benefit of the firm to be maximum;

b) The firm must modify the disposable quantity and it has two variants b_1, b_2. It must be chosen the optimum solution regarding the profit maximization.

c) Following some innovations, the products unitary profits modify and become 8, 6, respectively 7. Calculate the optimal solution given the new conditions. Can this variant be accepted, if the company must manufacture all three products without exception?

d) The company involves in manufacturing two products, P_4, P_5, with the unitar consumption according to the table. The benefit brought by P_4 is 3 and the one brought by P_5 is 5. Find out the maximum profit.

	P_1	P_2	P_3	Disp.	b_1	b_2	P_4	P_5
S_1	1	2	1	2400	2600	1600	2	3
S_2	1	0	3	800	600	1000	0	1
S_3	2	1	2	1000	1200	800	1	0
S_4	0	2	3	1200	1600	1200	0	1
Profit	3	4	5				3	5
Profit	8	6	7					

Solution:

a) The problem's mathematical model is:

$\max f(x) = 3x_1 + 4x_2 + 5x_3$

$$\begin{cases} x_1 + 2x_2 + x_3 \le 1200 \\ x_1 + x_3 \le 1400 \\ 2x_1 + x_2 + 2x_3 \le 2000 \\ 2x_2 + 3x_3 \le 800 \end{cases}$$

$x_i \ge 0,\ i = \overline{1,3}$

By rewriting the problem in standard form we get:

$\max f(x) = 3x_1 + 4x_2 + 5x_3 + 0(y_1 + y_2 + y_3 + y_4)$

$$\begin{cases} x_1 + 2x_2 + x_3 + y_1 = 1200 \\ x_1 + x_3 + y_2 = 1400 \\ 2x_1 + x_2 + 2x_3 + y_3 = 2000 \\ 2x_2 + 3x_3 + y_4 = 800 \end{cases}$$

$x_i \geq 0,\ i = \overline{1,3}$

$y_j \geq 0, j = \overline{1,4}$

The matrix of the equations system is:

$$\overline{A} = \begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The simplex table will be:

B	C_B	b	x_1	x_2	x_3	y_1	y_2	y_3	y_4	θ_{min}
			3	4	5	0	0	0	0	
y_1	0	1200	1	2	1	1	0	0	0	1200
y_2	0	1400	1	0	3	0	1	0	0	1400/3
y_3	0	2000	2	1	2	0	0	1	0	1000
y_4	0	800	0	2	3	0	0	0	1	800/3
	$\Delta_j = z_j^B - c_j$	0	-3	-4	-5	0*	0*	0*	0*	
y_1	0	2800/3	1	4/3	0	1	0	0	-1/3	2800/3
y_2	0	600	**1**	-2	0	0	1	0	-1	600
y_3	0	4400/3	2	-1/3	0	0	0	1	-2/3	2200/3
x_3	5	800/3	0	2/3	1	0	0	0	1/3	-
	$\Delta_j = z_j^B - c_j$	4000/3	-3	-2/3	0*	0*	0*	0*	5/3	
y_1	0	1000/3	0	10/3	0	1	-1	0	2/3	100
x_1	3	600	1	-2	0	0	1	0	-1	-
y_3	0	800/3	0	**11/3**	0	0	-2	1	4/3	800/11
x_3	5	800/3	0	2/3	1	0	0	0	1/3	400
	$\Delta_j = z_j^B - c_j$	9400/3	0*	-20/3	0*	0*	3	0*	-4/3	
y_1	0	1000/11	0	0	0	1	**9/11**	-10/11	-6/11	1000/9
x_1	3	8200/11	1	0	0	0	-1/11	6/11	-3/11	-
x_2	4	800/11	0	1	0	0	-6/11	3/11	4/11	-
x_3	5	2400/11	0	0	1	0	4/11	-2/11	1/11	600
	$\Delta_j = z_j^B - c_j$	39800/11	0*	0*	0*	0*	-7/11	20/11	12/11	

y_2	0	1000/9	0 0 0	11/9 1 -10/9 -6/9	
x_1	3	6800/9	1 0 0	1/9 0 4/9 -3/9	
x_2	4	1200/9	0 1 0	6/9 0 -3/9 0	
x_3	5	1600/9	0 0 1	-4/9 0 2/9 3/9	
	$\Delta_j = z_j^B - c_j$	33200/9	0* 0* 0*	7/9 0* 10/9 6/9	

B^{-1}

Therefore, for the company to obtain maximum profit, it is necessary to produce P_1, P_2 and P_3 in the following quantities:

$x_1 = \frac{6800}{9}$, $x_2 = \frac{1200}{9}$, $x_3 = \frac{1600}{9}$.

The maximum profit is:

$$\max f = C_B \cdot b = 0 \cdot \frac{1000}{9} + 3 \cdot \frac{6800}{9} + 4 \cdot \frac{1200}{9} + 5 \cdot \frac{1600}{9} = \frac{33200}{9}.$$

b) For the b_1 variant we have:

$$x'_B = B^{-1} \cdot b_1 = \begin{pmatrix} 11/9 & 1 & -10/9 & -6/9 \\ 1/9 & 0 & 4/9 & -3/9 \\ 6/9 & 0 & -3/9 & 0 \\ -4/9 & 0 & 2/9 & 3/9 \end{pmatrix} \cdot \begin{pmatrix} 1300 \\ 1200 \\ 1800 \\ 1000 \end{pmatrix} = \begin{pmatrix} 1100/9 \\ 5500/9 \\ 2400/9 \\ 1400/9 \end{pmatrix}$$

The solution x'_B is optimal, hence the company achieves maximum profit:

$$\max f = 0 \cdot \frac{1100}{9} + 3 \cdot \frac{5500}{9} + 4 \cdot \frac{2400}{9} + 5 \cdot \frac{1400}{9} = \frac{33100}{9}.$$

In case of b_2 variant we have:

$$x''_B = B^{-1} \cdot b_2 = \begin{pmatrix} 11/9 & 1 & -10/9 & -6/9 \\ 1/9 & 0 & 4/9 & -3/9 \\ 6/9 & 0 & -3/9 & 0 \\ -4/9 & 0 & 2/9 & 3/9 \end{pmatrix} \cdot \begin{pmatrix} 1500 \\ 1300 \\ 2400 \\ 800 \end{pmatrix} = \begin{pmatrix} -600/9 \\ 8700/9 \\ 1800/9 \\ 1200/9 \end{pmatrix}.$$

The solution not being admissible, we apply the dual simplex algorithm:

B	C_B	b	3 4 5 x_1 x_2 x_3	0 0 0 0 y_1 y_2 y_3 y_4
y_2	0	-600/9	0 0 0	11/9 1 **-10/9** -6/9
x_1	3	8700/9	1 0 0	1/9 0 4/9 -3/9
x_2	4	1800/9	0 1 0	6/9 0 -3/9 0
x_3	5	1200/9	0 0 1	-4/9 0 2/9 3/9
	$\Delta_j = z_j - c_j$	39300/9	0* 0* 0*	7/9 0* 10/9 6/9
y_3	0	60	0 0 0	-11/10 -9/10 1 6/10
x_1	3	940	1 0 0	6/10 4/10 0 -6/10
x_2	4	220	0 1 0	3/10 -3/10 0 2/10
x_3	5	120	0 0 1	-2/10 2/10 0 2/10
	$\Delta_j = z_j - c_j$	4300	0* 0* 0*	2 1 0* 0

The maximum profit corresponding to the b_2 variant is:

$$\max f = 0 \cdot 60 + 3 \cdot 940 + 4 \cdot 220 + 5 \cdot 120 = 4300$$

The company will choose the solution for which the profit is higher, namely the b_2 variant, case in which the optimal programme is:

$x_1 = 940$, $x_2 = 220$, $x_3 = 120$.

c) We modify the costs in the last simplex table and use the primal simplex algorithm

B	C_B	b	8 x1	6 x2	7 x3	0 y1	0 y2	0 y3	0 y4	θ_{min}
y2	0	1000/9	0	0	0	11/9	1	-10/9	6/9	-
x1	6	6800/9	1	0	0	1/9	0	4/9	-3/9	-
x2	7	1200/9	0	1	0	6/9	0	-3/9	0	-
x3	4	1600/9	0	0	1	-4/9	0	2/9	**3/9**	1600/3
	$\Delta_j = z_j^B - c_j$	72800/9	0*	0*	0*	0*	-1	9	2	
y2	0	1400/3	0	0	2	1/3	1	-2/3	0	
x1	8	2800/3	1	0	1	-1/3	0	2/3	0	
x2	6	400/3	0	1	0	2/3	0	-1/3	0	
y4	0	1600/3	0	0	3	-4/3	0	2/3	1	
	$\Delta_j = z_j^B - c_j$	24800/3	0*	0*	1	4/3	0*	10/3	0*	

In order to be achieved maximum profit, only two of the three problems are manufactured, namely P_1 in the amount of $\frac{2800}{3}$ and P_2 in the amount of $\frac{400}{3}$.

Therefore, this variant is not accepted in the imposed conditions.

d) We have:

$$a_4^B = B^{-1} \cdot a_4 = \begin{pmatrix} 11/9 & 1 & -10/9 & -6/9 \\ 1/9 & 0 & 4/9 & -3/9 \\ 6/9 & 0 & -3/9 & 0 \\ -4/9 & 0 & 2/9 & 3/9 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 7/9 \\ -10/9 \\ 12/9 \\ 4/9 \end{pmatrix}$$

$$a_5^B = B^{-1} \cdot a_5 = \begin{pmatrix} 11/9 & 1 & -10/9 & -6/9 \\ 1/9 & 0 & 4/9 & -3/9 \\ 6/9 & 0 & -3/9 & 0 \\ -4/9 & 0 & 2/9 & 3/9 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7/9 \\ -1/9 \\ 3/9 \\ 4/9 \end{pmatrix}$$

B	C_B	b	3 x_1	4 x_2	5 x_3	0 y_1	0 y_2	0 y_3	0 y_4	3 x_4	5 x_5	θ_{min}
y_2	0	1000/9	0	0	0	11/9	1	-10/9	-6/9	7/9	**7/9**	1000/7
x_1	3	6800/9	1	0	0	1/9	0	4/9	-3/9	-10/9	-1/9	-
x_2	4	1200/9	0	1	0	6/9	0	-3/9	0	12/9	3/9	400
x_3	5	1600/9	0	0	1	-4/9	0	2/9	3/9	4/9	4/9	400
	$\Delta_j = z_j^B - c_j$	33200/9	0*	0*	0*	7/9	0*	10/9	6/9	11/9	-16/9	
x_5	5	1000/7	0	0	0	11/7	9/7	-10/7	-6/7	1	1	-
x_1	3	5400/7	1	0	0	2/7	1/7	2/7	-3/7	-1	0	-
x_2	4	600/7	0	1	0	1/7	-3/7	1/7	2/7	1	0	300
x_3	5	800/7	0	0	1	-8/7	-4/7	6/7	**5/7**	0	0	160

	$\Delta_j = z_j^B - c_j$	27600/7	0*	0*	0*	25/7	16/7	-10/7	-12/7	3	0	
x_5	5	280	0	0	6/5	1/5	3/5	-2/5	0	1	1	-
x_1	3	840	1	0	3/5	-2/5	-1/5	4/5	0	-1	0	1050
x_2	4	40	0	1	-2/5	3/5	-1/5	-1/5	0	1	0	-
y_4	0	160	0	0	7/5	-8/5	-4/5	**6/5**	1	0	0	400/3
	$\Delta_j = z_j^B - c_j$	4080	0*	0*	6/5	11/5	8/5	-2/5	0*	3	0	
x_5	5	2000/6	0	0	10/6	-2/6	2/6	0	2/6	1	1	
x_1	3	4400/6	1	0	-2/6	4/6	2/6	0	-4/6	-1	0	
x_2	4	400/6	0	1	-1/6	2/6	-2/6	0	1/6	1	0	
y_3	0	800/6	0	0	7/6	-8/6	-4/6	1	1/6	0	0	
	$\Delta_j = z_j^B - c_j$	24800/6	0*	0*	10/6	10/6	8/6	0*	2/6	3	0	

The maximum profit is:

$$\max f = 5 \cdot \frac{2000}{6} + 3 \cdot \frac{4400}{6} + 4 \cdot \frac{400}{6} + 0 \cdot \frac{800}{6} = \frac{24800}{6}$$

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